Diffusion effects in a nonlinear electrical lattice

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We consider a nonlinear electrical network modeling the generalized Nagumo equation. Focusing on the particular case where the initial load of the lattice consists in the superimposition of a coherent information weakly varying in space and a perturbation of small amplitude, we show that the perturbation can be eliminated quickly, almost without disturbing the information.

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Electrical transmission lines are very convenient tools for studying quantitatively the fascinating properties of nonlinear waves [1,2]. Indeed, they provide a useful way to check directly how the nonlinear excitations behave inside the medium, by means of probes related to an oscilloscope. The propagation of Korteweg–de Vries (KdV) solitons [3], of nonlinear Schrödinger (NLS) modulated waves [4], or the influence of discreteness on the modulational instability [5] are some recent examples of experimental studies devoted to nonlinear phenomena, using electrical lattices.

On the other hand, systems modeled by nonlinear reaction-diffusion equations have been widely investigated because of their importance in various fields like chemistry, biology, or ecology [6]. Since the pioneering study of Hodgkin and Huxley 45 years ago [7], a special interest has been devoted to the understanding of the transmission of information along nerve fibers [8]. In particular, Nagumo [9] has realized an experimental electrical lattice using tunnel diodes for simulating the propagation of pulses in nerve axon. The effects of discreteness, which may radically change the propagation or diffusion characteristics with regards to the case of a continuous system, have also been discussed [10,11] to explain the so-called propagation failure in the Nagumo equation.

Reaction-diffusion phenomena can also be applied in the field of signal processing. Indeed, Chua introduced cellular neural networks (CNN) [12], which represent, under certain conditions [13], an excellent approximation to the nonlinear partial differential equations describing reaction-diffusion systems, such as Fisher's equation, FitzHugh-Nagumo equation, or the generalized Nagumo equation. These CNNs are inspired by the biological neural networks and are capable of high-speed parallel signal or image processing [14].

In this paper, we present a one-dimensional reaction-diffusion electrical lattice (1D CNN) modeling the generalized Nagumo equation and allowing noise removal. In the particular case where the initial load of the lattice consists in the superimposition of a coherent information weakly varying in space and a perturbation of small amplitude, we show that the perturbation can be eliminated quickly, almost without disturbing the information.

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We consider a nonlinear diffusive electrical lattice made of N identical cells, as illustrated in Fig. 1. Each cell contains a series linear resistor R_1 and a linear capacitor C in parallel with a nonlinear resistor R_{NL}. From Kirchhoff laws, we derive the system of nonlinear discrete equations for $2 \leqslant n \leqslant N-1$:

$$\frac{dV_n}{dt} = \frac{1}{R_1 C}\,(V_{n+1}+V_{n-1}-2V_n) - \frac{g(V_n)}{C}, \tag{1}$$

where V_n and $g(V_n)=I_{\mathrm{NL},n}$ are respectively the voltage at cell n and the corresponding nonlinear current in the resistor R_{NL}. The description of the system is completed by assuming zero-flux or Neumann boundary conditions, for $n=1$ and $n=N$, respectively,

$$\frac{dV_1}{dt} = \frac{1}{R_1 C}\,(V_2-V_1) - \frac{g(V_1)}{C}, \tag{2a}$$

$$\frac{dV_N}{dt} = \frac{1}{R_1 C}\,(V_{N-1}-V_N) - \frac{g(V_N)}{C}. \tag{2b}$$

Equation (1) is the discrete version of a diffusion equation introduced by Nagumo [9] for simulating information propagation in nerve axon

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v), \tag{3}$$

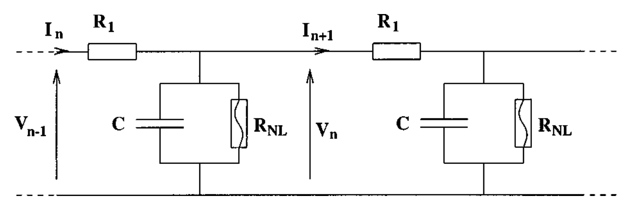

FIG. 1. Schematic representation of the nonlinear diffusive electrical network. The lattice is composed of $N=256$ identical cells with a linear resistance R_1 and a nonlinear resistance R_{NL} connected in parallel with a linear capacitor C.

in which $f(v)$ is a cubic polynomial. Here, the current-voltage characteristics $I_{\mathrm{NL},n}=g(V_n)$, similar to that of a Gunn diode, can be expressed by the following cubic function:

$$I_{\mathrm{NL},n}=g(V_n)=\frac{V_n}{R_0}\left(1-\frac{V_n}{V_a}\right)\left(1-\frac{V_n}{V_b}\right). \tag{4}$$

In this expression, V_a and V_b are constant voltages and obey $2V_a \leqslant V_b$, while R_0 is the value of the nonlinear resistor R_{NL} in the linear regime, e.g., for $V_n \ll V_a < V_b$. Furthermore, when a uniform excitation is applied in the lattice [$V_{n+1}+V_{n-1}-2V_n=0$ in Eq. (1)], it is known that [6], the two steady states 0 and V_b are stable, while steady state V_a is unstable. Finally, in the following, we restrict our study to the case where $V_n \in [0,V_b]$.

We focus now on the more interesting case where the network is initially loaded with a nonuniform signal consisting of the superimposition of a coherent voltage (the ''information''), and a perturbation, e.g., white noise, of small amplitude. The coherent part of the signal is supposed to vary slowly in time and space compared to the perturbation. Therefore, it will be treated as a continuous signal and we introduce the slow variables $X=\varepsilon^{\sqrt{\alpha}}n$ and $\tau=\varepsilon^{\alpha}t$, where ε is a small parameter and $\alpha \to 0$. Then, the resulting voltage at cell n can be expressed by

$$V_n(t)=U(X,\tau)+\varepsilon^{\beta}b_n(t), \tag{5}$$

where $U(X,\tau)$ represents the coherent part of the signal, while $\varepsilon^{\beta}b_n(t)$ is the perturbation. Note that coefficient β is assumed to be greater than α in order to express that the amplitude of the perturbation is small compared to the amplitude of the coherent signal. Using the reductive perturbation method [15], we insert expression (5) in (1), and, collecting terms of order lower than ε^{β}, we find that the voltage U obeys the continuous Nagumo equation

$$\frac{\partial U}{\partial \tau}=\frac{1}{\tau_1}\frac{\partial^2 U}{\partial X^2}-\frac{1}{\tau_2}U\left(1-\frac{U}{V_a}\right)\left(1-\frac{U}{V_b}\right), \tag{6}$$

where $\tau_1=1/R_1C$ and $\tau_2=1/R_0C$. As the voltage U is assumed to be slowly varying in space, we can ignore, in Eq. (6), the space gradient term with respect to the nonlinear term. Then, in the case where $V_a=V_b/2$, which maintains the symmetry of the device, we obtain analytically the evolution of U versus time:

$$U(t)=\frac{V_b}{2}\left[1+\frac{(U_0-V_b/2)}{\sqrt{(U_0-V_b/2)^2+U_0(V_b-U_0)e^{-t/\tau_2}}}\right], \tag{7}$$

where U_0 is the initial condition. As a result, two cases of evolution are possible, depending on the value of U_0: (i) When $0 \leqslant U_0 < V_b/2$, the voltage in the lattice tends uniformly to the stable state 0; (ii) when $V_b/2 < U_0 \leqslant V_b$, the voltage in the lattice tends uniformly to the stable state V_b. Notice that the special case where $U_0=V_b/2$, which corresponds to an unstable state, is not considered.

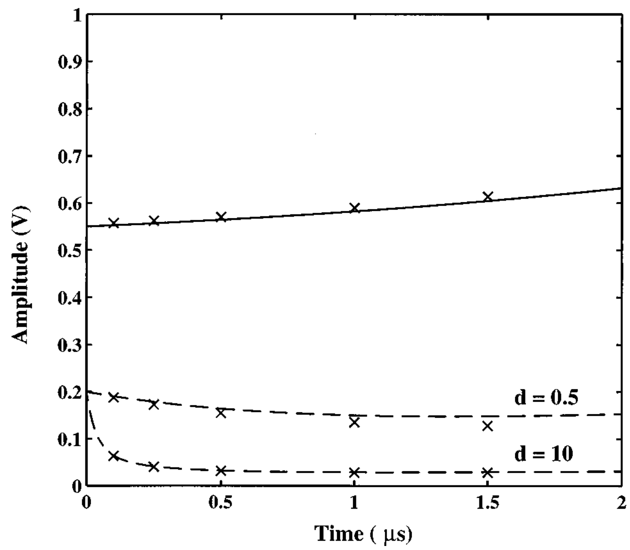

FIG. 2. Theoretical evolution of the noise vs time for $d=\tau_2/\tau_1=10$ and $d=0.5$ (dashed lines). The theoretical evolution of the information signal (continuous line) remains identical for the two cases $d=10$ and $d=0.5$. Results obtained by numerical simulations are represented by $\times$ signs.

Next, collecting terms of order ε^{β} in Eq. (1), we obtain the linear discrete equation that governs the evolution of a perturbation of small amplitude in the lattice

$$\frac{db_n}{dt}=\frac{1}{\tau_1}[b_{n+1}+b_{n-1}-2b_n] -\frac{1}{\tau_2}b_n\left[1-2U\left(\frac{1}{V_a}+\frac{1}{V_b}\right)+\frac{3U^2}{V_aV_b}\right]. \tag{8}$$

As we know U at any time through Eq. (7), we can determine the evolution of the noise $b_n(t)$, that is, if $b_{n'}(0)$ is the initial small amplitude of the noise located at cell n':

$$b_n(t)=\left(\frac{V_b}{2}\right)^3\sum_{n'}\frac{b_{n'}(0)I_{n'-n}(2t/\tau_1)e^{(1/2\tau_2-2/\tau_1)t}}{[(U_0-V_b/2)^2e^{t/\tau_2}+U_0(V_b-U_0)]^{3/2}}, \tag{9}$$

where $I_{n'-n}(t)$ is the modified Bessel function of order $n'-n$. Equations (7) and (9) show that the time scales for the evolutions of both noise and coherent signal depend on the components of the elementary cell (see Fig. 1) by way of τ_1 and τ_2. For example, they can be chosen in order to speed up the noise diffusion without significantly disturbing the coherent part U of the signal.

In order to check the validity of our analytical approach, we have performed theoretical calculations and numerical simulations. The parameters of the nonlinear current-voltage characteristics have been chosen to be $V_b=1$ V and $V_a=V_b/2=0.5$ V.

Following our hypotheses for the analytical approach, we consider, for initial conditions, a single noisy impulsion of small amplitude located on cell $n=N/2$, that is $b_n(t=0)=0.2\delta_{n,N/2}$ V ($\delta_{i,j}$ being the Kronecker symbol), superimposed onto a uniform signal of amplitude $U_0=0.55$ V ($V_b/2<U_0<V_b$).

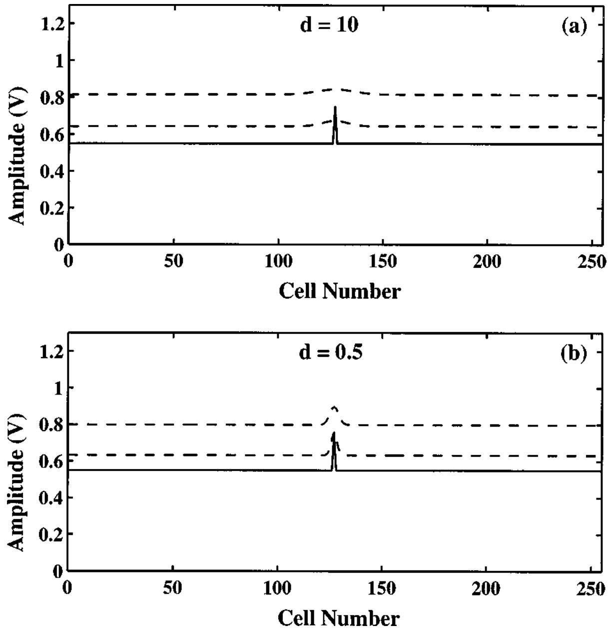

FIG. 3. Evolution of the uniform signal with a localized perturbation for $d=10$ (a) and $d=0.5$ (b). Signals are represented for both cases at $t=0$ s (continuous line), $t=2$ μs (dashed line), and $t=4$ μs (dashed line above).

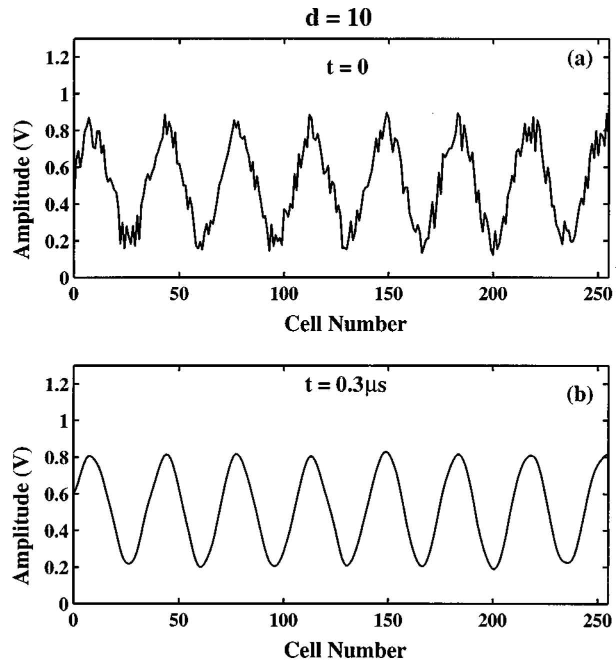

FIG. 4. Processing of a noisy periodic signal with $d=10$: (a) initial condition at $t=0$ s; (b) resulting signal at $t=0{,}3$ μs.

First, using Eqs. (7) and (9) for $\tau_1=0.1$ μs and $\tau_2=1$ μs, that is with a discretization parameter [11] $d=\tau_2/\tau_1=10$, we obtain the time evolutions of $U(t)$ and $b_n(t)$ represented in Fig. 2 by the solid and the dashed lines, respectively. In this case, the system behaves almost like a continuous one, and the noise tends quickly to zero while the information U evolves slowly and tends to stable state $V_b=1$ V. On the other hand, if $\tau_1=2$ μs and $\tau_2=1$ μs, that is, $d=0.5$, the noise evolution (upper dashed line in Fig. 2) in this rather discrete system is similar to the behavior of the information signal U, which remains identical to that of the previous case (continuous line). The case $\tau_1=0.1$ μs and $\tau_2=1$ μs, that is, $d=10$, seems then favorable for eliminating noise without perturbing too much the coherent part of the signal.

Next, numerical simulations of Eq. (1) have been performed on a lattice of $N=256$ cells, with Neumann boundary conditions and choosing both the favorable case $d=10$ and the case $d=0.5$. Furthermore, at $t=0$, the network is loaded with the initial condition depicted above and represented by the continuous line in Figs. 3(a) and 3(b), while the signal in the lattice is represented at times $t=2$ μs and $t=4$ μs by the dashed lines. As time evolves and according to Eqs. (7) and (9), for $d=10$ [see Fig. 3(a)] the single noise impulsion decreases quickly, while the coherent signal is slowly attracted by the nearest stable state, here $U=V_b=1$ V. On the contrary, for $d=0.5$, as presented in Fig. 3(b), the perturbation and the information follow the same evolution: no filtering of the noise occurs.

At this stage, it is necessary to compare quantitatively the results, given by the numerical simulations, to the theoretical predictions. During the simulations, the amplitudes of, respectively, the uniform signal and the perturbation, have been measured for five different values of time ($t=0.1$ μs, 0.25 μs, 0.5 μs, 1 μs, and 1.25 μs) and for both cases $d=10$ and $d=0.5$. For a direct comparison with the theoretical predictions, these amplitudes have been plotted in Fig. 2 ($\times$ signs), superimposed to the curves giving the predicted evolutions of $U(t)$ and $b_n(t)$. The agreement between our model and the simulations is quite satisfactory for the evolutions of both the coherent signal $U(t)$ on one hand, and the perturbation $b_n(t)$ on the other hand, even if there exists for $b_n(t)$ a slight discrepancy in the case $d=0.5$.

Finally, we present, in Fig. 4, the processing of a more interesting signal, that is, a noisy periodic signal, in the favorable case $d=10$. This signal consists in a sinusoid of amplitude $U=0.3$ V superimposed onto a white noise of amplitude 0.1 V [Fig. 4(a)]. As expected, the perturbation part of the whole signal disappears very quickly as shown in Fig. 4(b) for $t=0.3$ μs.

Our system, based on a nonlinear lattice modeling the Nagumo equation, could be useful for realizing an analog signal processing array allowing the improvement of the signal-to-noise ratio. An experimental electrical lattice is currently under investigation in order to check this theoretical study. Furthermore, it will allow us to observe on a real system the properties of the Nagumo equation (front propagation, discreteness effects, etc.).

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